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Integral equation for gauge invariant quark two-point Green's function in QCD

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Abstract

Gauge invariant quark two-point Green's functions defined with path-ordered gluon field phase factors along skew-polygonal lines joining the quark to the antiquark are considered. Functional relations between Green's functions with different numbers of path segments are established. An integral equation is obtained for the Green's function defined with a phase factor along a single straight line. The equation implicates an infinite series of two-point Green's functions, having an increasing number of path segments; the related kernels involve Wilson loops with contours corresponding to the skew-polygonal lines of the accompanying Green's function and with functional derivatives along the sides of the contours. The series can be viewed as an expansion in terms of the global number of the functional derivatives of the Wilson loops. The lowest-order kernel, which involves a Wilson loop with two functional derivatives, provides the framework for an approximate resolution of the equation.

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1 Introduction

Gauge invariant objects are expected to provide a more precise description of observable quantities than gauge variant ones. Generally, gauge invariance of multilocal operators is ensured with the use of path-ordered phase factors [1, 2]. In this respect, the closed loop operator, the so-called Wilson loop, showed itself a powerful tool for the investigation of the confinement properties of QCD [3, 4, 5]. The properties of the Wilson loop were studied in detail in a long series of papers [6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

On the other hand, the usual machinery of quantum field theory, based on the Dyson–Schwinger integral equations [16, 17], does not apply in a straightforward way to Green's functions of operators involving path-ordered phase factors. The main reason is related to the difficulty of obtaining the functional inverses of the nonlocal gauge invariant Green's functions and thus of being able to define analogues of proper vertices, which play a crucial role in the formulation of integral equations. Expressions of gauge invariant quark-antiquark Green's functions in terms of Wilson loops were obtained in the past [18, 19, 20] with the use of the Feynman–Schwinger representation of the quark propagator [21, 22, 23, 24]; these, however, could not be transformed into equivalent integral equations without the recourse to approximations related to the quark motion.

The purpose of the present paper is to investigate the possibilities of deriving integral or integro-differential equations for gauge invariant Green's functions which might allow for a systematic study of their various properties. We concentrate in this work on the quark gauge invariant two-point function, in which the quark and the antiquark fields are joined by a path-ordered phase factor, but the methods which we shall develop are readily applicable to more general cases.

Our starting point is a particular representation of the quark propagator in the presence of an external gluon field, where it is expressed as a series of terms involving path-ordered phase factors along successive straight lines forming generally skew-polygonal lines. That representation is a relativistic generalization of the one introduced by Eichten and Feinberg in the nonrelativistic case [25]; it was already used in a previous work for deriving a bound state equation for quark-antiquark systems [26]; however, in the latter work, the bound state equation was derived by circumventing the explicit writing of an integral equation for the related Green's function and of the neglected higher-order terms of the interaction kernel. One of the main properties of the above representation is that in gauge invariant quantities, at each order of the expansion, the paths of the phase factors close up to form a Wilson loop. Thus, the corresponding Green's function becomes expressed, through a series expansion, in terms, among others, of Wilson loops having skew-polygonal contours with an increasing number of sides.

Several differences occur with respect to the formulation of the Dyson–Schwinger equations. First, for the reasons mentioned above, proper vertices are not introduced; instead,

we work directly with Green's functions; the various kernels that appear are written explicitly in terms of functional derivatives of the logarithm of the Wilson loop average and of the quark Green's function. Second, starting from the simplest gauge invariant two-point function, constructed with a phase factor along a single straight line joining the quark to the antiquark, one generates, through the equations of motion, a chain of new gauge invariant two-point functions with phase factors along n-sided skew-polygonal lines between the quark and the antiquark ($n > 1$). On the other hand, every such Green's function (with n skew-polygonal sides) can be related with the aid of functional relations to the lowest-order Green's function ($n = 1$) and thus, in principle, an equation involving only the latter Green's function is possible to construct. The third difference arises at the level of the presence of nested kernels, which do not occur in the Dyson–Schwinger equations and which persist here due to background effects induced by the Wilson loops: each nested kernel is modified by its new background when inserted inside a higher-order Wilson loop as compared to its original expression. The remaining terms in the kernels have the property of conventional irreducibility.

The integral equation that we obtain is constructed as an expansion in terms of the global number of derivatives of the logarithm of the Wilson loop average. Although it involves an infinite series of kernels and Green's functions, at each order of the expansion the explicit expressions of the kernels and of the relations between high-order Green's functions with the lowest-order one can be obtained from definite formulas.

On practical grounds, an increasing number of derivatives of a Wilson loop, each derivative occurring on a different region of the contour, is generally expected to give a relatively decreasing contribution at short- and at large-distances. Therefore, the series expansion of the kernels in terms of functional derivatives of Wilson loops can also be considered as a perturbative expansion, the most important contribution coming from the lowest-order non-vanishing term and involving the smallest number of derivatives. That property allows us to consider solving the integral equation with appropriate approximations.

The plan of the paper is the following. In Sec. 2, we introduce the definitions and conventions that will be used throughout this work. Section 3 deals with the representation of the quark propagator in external field in terms of path-ordered phase factors. In Sec. 4, functional relations are established between various Green's functions. In Sec. 5, the integral equation for the gauge invariant quark two-point function with a straight line path is established and the structure of the kernel terms is displayed. Section 6 deals with the question of analyticity properties of the Green's function. A summary and comments follow in Sec 7. Two appendices are devoted to the presentation of the summation method with free propagators and the study of the self-energy function.

2 Definitions and conventions

We introduce in this section the main definitions and conventions that we shall use throughout this work.

We consider a path-ordered phase factor along a line C_{yx} joining a point x to a point y, with an orientation defined from x to y:

$$U(C_{yx};y,x) \equiv U(y,x) = Pe^{-ig\int_x^y dz^\mu A_\mu(z)}, \tag{2.1}$$

where $A_\mu = \sum_a A_\mu^a t^a$, A_μ^a $(a = 1,\ldots,N_c^2-1)$ being the gluon fields and t^a the generators of the gauge group $SU(N_c)$ in the fundamental representation, with the normalization $\mathrm{tr}t^at^b = \frac{1}{2}\delta^{ab}$. A more detailed definition of U is given by the series expansion in the coupling constant g; all equations involving U can be obtained from the latter expression. Parametrizing the line C with a parameter λ, $C = \{x(\lambda)\}$, $0 \leq \lambda \leq 1$, such that $x(0) = x$ and $x(1) = y$, a variation of C induces the following variation of U $[U(x(\lambda),x(\lambda')) \equiv U(\lambda,\lambda')$, $A(x(\lambda)) \equiv A(\lambda)]$:

$$\begin{aligned}\delta U(1,0) &= -ig\delta x^\alpha(1)A_\alpha(1)U(1,0) + igU(1,0)A_\alpha(0)\delta x^\alpha(0)\\ &\quad +ig\int_0^1 d\lambda U(1,\lambda)x'^\beta(\lambda)F_{\beta\alpha}(\lambda)\delta x^\alpha(\lambda)U(\lambda,0),\end{aligned} \tag{2.2}$$

where $x' = \frac{\partial x}{\partial\lambda}$ and F is the field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. The variations inside the integral lead to functional differentiation of U, while the variations at the end points (marked points) are defined as leading to ordinary differentiation. The functional derivative of U with respect to $x(\lambda)$ $(0 < \lambda < 1)$ is then [6]:

$$\frac{\delta U(1,0)}{\delta x^\alpha(\lambda)} = igU(1,\lambda)x'^\beta(\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0). \tag{2.3}$$

For paths defined along rigid lines, the variations inside the integral in Eq. (2.2) are related, with appropriate weight factors, to those of the end points; a displacement of one end point generates a displacement of the whole line with the other end point left fixed. Considering now a rigid straight line between x and y, an ordinary derivation at the end points yields:

$$\frac{\partial U(y,x)}{\partial y^\alpha} = -igA_\alpha(y)U(y,x) + ig(y-x)^\beta \int_0^1 d\lambda\,\lambda\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0), \tag{2.4}$$

$$\frac{\partial U(y,x)}{\partial x^\alpha} = +igU(y,x)A_\alpha(x) + ig(y-x)^\beta \int_0^1 d\lambda\,(1-\lambda)\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0). \tag{2.5}$$

When considering path variations of gauge invariant quantities, with paths made of segments, the end point contributions involving the explicit A_α terms disappear, being

cancelled by similar contributions coming from neighboring segments or from variations of neighboring fields. The general contributions that remain at the end are those coming from the internal part of the segments represented by the integrals in Eqs. (2.4)-(2.5). We adopt the following conventions to represent such contributions:

$$\frac{\bar{\delta}U(y,x)}{\bar{\delta}y^{\alpha+}} \equiv ig(y-x)^{\beta}\int_0^1 d\lambda\,\lambda\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0), \tag{2.6}$$

$$\frac{\bar{\delta}U(y,x)}{\bar{\delta}x^{\alpha-}} \equiv ig(y-x)^{\beta}\int_0^1 d\lambda\,(1-\lambda)\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0). \tag{2.7}$$

The first equation above corresponds to a displacement of the end point of the segment (taking into account the orientation on the path), while the second equation corresponds to a displacement of the starting point of the segment. Equations (2.4) and (2.5) can be written as

$$\frac{\partial U(y,x)}{\partial y^{\alpha}} = -igA_{\alpha}(y)U(y,x) + \frac{\bar{\delta}U(y,x)}{\bar{\delta}y^{\alpha+}}, \tag{2.8}$$

$$\frac{\partial U(y,x)}{\partial x^{\alpha}} = +igU(y,x)A_{\alpha}(x) + \frac{\bar{\delta}U(y,x)}{\bar{\delta}x^{\alpha-}}. \tag{2.9}$$

If two phase factors $U(z,y)$ and $U(y,x)$ along segments are joined at the point y (a marked point), then, with the aid of the previous notations, we have:

$$\frac{\partial}{\partial y^{\alpha}}\Big(U(z,y)U(y,x)\Big) = \frac{\bar{\delta}U(z,y)}{\bar{\delta}y^{\alpha-}}U(y,x) + U(z,y)\frac{\bar{\delta}U(y,x)}{\bar{\delta}y^{\alpha+}}. \tag{2.10}$$

The Wilson loop, denoted $\Phi(C)$, is defined as the trace in color space of the path-ordered phase factor (2.1) along a closed contour C:

$$\Phi(C) = \frac{1}{N_c}\mathrm{tr}Pe^{-ig\oint_C dx^{\mu}A_{\mu}(x)}, \tag{2.11}$$

where the factor $1/N_c$ has been put for normalization. It is a gauge invariant quantity. Its vacuum expectation value is denoted $W(C)$:

$$W(C) = \langle\Phi(C)\rangle, \tag{2.12}$$

the averaging being defined in the path integral formalism.

We shall represent the Wilson loop average as an exponential function, whose argument is a functional of the contour C [7, 14]:

$$W(C) = e^{F(C)}. \tag{2.13}$$

In perturbation theory, $F(C)$ is given by the sum of all connected diagrams, the connection being defined with respect to the contour C, after subtraction of reducible parts [14].

Variations of $W(C)$ due to local deformations of the contour C can then be expressed in terms of variations of $F(C)$:

$$\frac{\delta W(C)}{\delta x^{\alpha}}\bigg|_{x\in C} = \frac{\delta F(C)}{\delta x^{\alpha}}\bigg|_{x\in C} W(C). \tag{2.14}$$

This property is also generalized to the case of rigid variations of paths (segments). If the contour C is a skew-polygon C_n with n sides and n successive marked points x_1, x_2, $\ldots$, x_n at the cusps, then we write:

$$W(x_n, x_{n-1}, \ldots, x_1) = W_n = e^{F_n(x_n, x_{n-1}, \ldots, x_1)} = e^{F_n}, \tag{2.15}$$

the orientation of the contour going from x_1 to x_n through x_2, x_3, etc. (i.e., towards xs with indices increasing by one unit). Then, according to the definitions (2.6)-(2.10), the notation $\bar{\delta}F_n/\bar{\delta}x_i^-$ means that the derivation acts on the internal part of the segment x_ix_{i+1} with x_{i+1} held fixed ($x_{n+1} = x_1$), while $\bar{\delta}F_n/\bar{\delta}x_i^+$ means that the derivation acts on the internal part of the segment $x_{i-1}x_i$ with x_{i-1} held fixed ($x_0 = x_n$).

The gauge invariant two-point quark Green's function is defined as

$$S_{\alpha\beta}(x, x'; C_{x'x}) = -\frac{1}{N_c}\,\langle\overline{\psi}_\beta(x')\,U(C_{x'x}; x', x)\,\psi_\alpha(x)\rangle, \tag{2.16}$$

α and β being the Dirac spinor indices, while the color indices are implicitly summed. In the present work we shall mainly deal with paths along skew-polygonal lines. For such lines with n sides and $n-1$ junction points y_1, y_2, $\ldots$, y_{n-1} between the segments, we define:

$$S_{(n)}(x, x'; y_{n-1}, \ldots, y_1) = -\frac{1}{N_c}\,\langle\overline{\psi}(x')U(x', y_{n-1})U(y_{n-1}, y_{n-2})\ldots U(y_1, x)\psi(x)\rangle. \tag{2.17}$$

The simplest such function corresponds to $n = 1$, for which the points x and x' are joined by a single straight line:

$$S_{(1)}(x, x') \equiv S(x, x') = -\frac{1}{N_c}\,\langle\overline{\psi}(x')\,U(x', x)\,\psi(x)\rangle. \tag{2.18}$$

(We shall generally omit the index 1 from that function.)

The free propagator will be designated by S_0 (without color group content):

$$S_0(x, x') = S_0(x - x') = \int \frac{d^4p}{(2\pi)^4}\,e^{-ip.(x - x')}\,\frac{i}{\gamma.p - m + i\varepsilon}. \tag{2.19}$$

In conjunction with the definitions (2.6)-(2.7), we shall also introduce the notations

$$\frac{\bar{\delta}S(x, x')}{\bar{\delta}x^{\mu-}} = -\frac{1}{N_c}\,\langle\overline{\psi}(x')\,\frac{\bar{\delta}U(x', x)}{\bar{\delta}x^{\mu-}}\,\psi(x)\rangle, \quad \frac{\bar{\delta}S(x, x')}{\bar{\delta}x'^{\nu+}} = -\frac{1}{N_c}\,\langle\overline{\psi}(x')\,\frac{\bar{\delta}U(x', x)}{\bar{\delta}x'^{\nu+}}\,\psi(x)\rangle. \tag{2.20}$$

3 The quark propagator in external field

We shall use a two-step quantization method, by first integrating the quark fields and then, at a second stage, integrating the gluon fields through Wilson loops. The first operation yields among various quantities the quark propagator in the presence of an arbitrary external gluon field. The latter, designated by $S(x,x';A)$, or by $S(A)$ for short, satisfies the usual equation

$$\Big(i\gamma.\partial_{(x)} - m - g\gamma.A(x)\Big)S(x,x';A) = i\delta^4(x-x'). \tag{3.1}$$

To exhibit a Wilson loop structure in gauge invariant quantities, it is necessary to describe the quark propagator in external field by means of path-ordered phase factors. To this end, we shall first introduce a representation, already used in Ref. [26], which combines path-ordered phase factors along straight lines and free quark propagators. At a later stage, to sum directly self-energy effects, we shall replace the free quark propagator by the full gauge invariant Green's fuction (2.18).

The starting point of the representation is the gauge covariant composite object, denoted $\widetilde{S}_0(x,x')$, made of a free fermion propagator $S_0(x,x')$ (without color group content) multiplied by the path-ordered phase factor $U(x,x')$ [Eq. (2.1)] taken along the straight line xx':

$$\Big[\widetilde{S}_0(x,x')\Big]^a_{\ b} \equiv S_0(x,x')\Big[U(x,x')\Big]^a_{\ b}. \tag{3.2}$$

[a,b: color indices.] The advantage of the straight line over other types of line is that under Lorentz transformations it remains form invariant and in the limit $x' \to x$ U tends to unity in an unambiguous way. $\widetilde{S}_0$ satisfies the following equation with respect to x:

$$\Big(i\gamma.\partial_{(x)} - m - g\gamma.A(x)\Big)\widetilde{S}_0(x,x') = i\delta^4(x-x') + i\gamma^\alpha \frac{\bar{\delta}U(x,x')}{\bar{\delta}x^{\alpha+}} S_0(x,x'). \tag{3.3}$$

A similar equation also holds with respect to x', with x held fixed, with the Dirac and color group matrices acting from the right.

The quantity $-i(i\gamma.\partial_{(x)} - m - g\gamma.A(x))\delta^4(x-x')$ is the inverse of the quark propagator $S(x,x';A)$ in the presence of the external gluon field A. Reversing Eq. (3.3) with respect to $S(A)^{-1}$, one obtains an equation for $S(A)$ in terms of $\widetilde{S}_0$:

$$S(x,x';A) = \widetilde{S}_0(x,x') - \int d^4x''\, S(x,x'';A)\,\gamma^\alpha\, \frac{\bar{\delta}\widetilde{S}_0(x'',x')}{\bar{\delta}x''^{\alpha+}}. \tag{3.4}$$

Using the equation with x', or making in Eq. (3.4) an integration by parts, one obtains another equivalent equation:

$$S(x,x';A) = \widetilde{S}_0(x,x') + \int d^4x''\, \frac{\bar{\delta}\widetilde{S}_0(x,x'')}{\bar{\delta}x''^{\alpha-}}\,\gamma^\alpha\, S(x'',x';A). \tag{3.5}$$

Equations (3.4) or (3.5) allow us to obtain the propagator $S(A)$ as an iteration series with respect to $\widetilde{S}_0$, which contains the free fermion propagator, by maintaining at each

order of the iteration its gauge covariance property. For instance, the expansion of Eq. (3.4) takes the form:

$$S(x,x';A) \;=\; \widetilde{S}_0(x,x') - \int d^4y_1\, \widetilde{S}_0(x,y_1)\,\gamma^{\alpha_1}\,\frac{\bar{\delta}\widetilde{S}_0(y_1,x')}{\bar{\delta}y_1^{\alpha_1+}}$$
$$+\int d^4y_1 d^4y_2\, \widetilde{S}_0(x,y_1)\,\gamma^{\alpha_1}\,\frac{\bar{\delta}\widetilde{S}_0(y_1,y_2)}{\bar{\delta}y_1^{\alpha_1+}}\,\gamma^{\alpha_2}\,\frac{\bar{\delta}\widetilde{S}_0(y_2,x')}{\bar{\delta}y_2^{\alpha_2+}} + \cdots\ . \tag{3.6}$$

Equations (3.4) and (3.5) are relativistic generalizations of the representation used for heavy quark propagators starting from the static case [25].

In order to sum, for later purposes, self-energy effects, one can use for the expansion of the propagator $S(A)$, instead of the free propagator S_0, the full gauge invariant Green's function (2.18). To this end, we define a generalized version of the gauge covariant object $\widetilde{S}_0$ [Eq. (3.2)], by replacing in it S_0 with S [Eq. (2.18)]:

$$\Big[\widetilde{S}(x,x')\Big]^a_{\ b} \equiv S(x,x')\Big[U(x,x')\Big]^a_{\ b}. \tag{3.7}$$

The Green's function S satisfies the following equations of motion:

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu\,\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}}, \tag{3.8}$$

$$S(x,x')(-i\gamma.\overleftarrow{\partial}_{(x')} - m) = i\delta^4(x-x') - i\frac{\bar{\delta}S(x,x')}{\bar{\delta}x'^{\mu+}}\,\gamma^\mu. \tag{3.9}$$

[Notice that the orientation of the path in $S(x,x')$ is from x to x'.] Then $\widetilde{S}$ satisfies the equation

$$\Big(i\gamma.\partial_{(x)} - m - g\gamma.A(x)\Big)\widetilde{S}(x,x') = i\delta^4(x-x')$$
$$+i\gamma^\alpha\Big(\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\alpha-}}U(x,x') + S(x,x')\frac{\bar{\delta}U(x,x')}{\bar{\delta}x^{\alpha+}}\Big), \tag{3.10}$$

from which one deduces the expansion of $S(A)$ around S:

$$S(x,x';A) = S(x,x')U(x,x') - S(x,y;A)\,\gamma^\alpha\,\Big(\frac{\bar{\delta}S(y,x')}{\bar{\delta}y^{\alpha-}}U(y,x') + S(y,x')\frac{\bar{\delta}U(y,x')}{\bar{\delta}y^{\alpha+}}\Big). \tag{3.11}$$

[The integrations on intermediate points are implicit.] Using the equations of S and $S(A)$ relative to x', or making in Eq. (3.11) an integration by parts, one obtains another equivalent equation:

$$S(x,x';A) = S(x,x')U(x,x') + \Big(\frac{\bar{\delta}S(x,y)}{\bar{\delta}y^{\alpha+}}U(x,y) + S(x,y)\frac{\bar{\delta}U(x,y)}{\bar{\delta}y^{\alpha-}}\Big)\,\gamma^\alpha\, S(y,x';A). \tag{3.12}$$

A graphical representation of Eq. (3.12) is shown in Fig. 1

Equations (3.11)-(3.12) constitute the basic formulas that will be used to express equations of motion of gauge invariant quark Green's functions in terms of Wilson loops and gauge invariant two-point Green's functions.

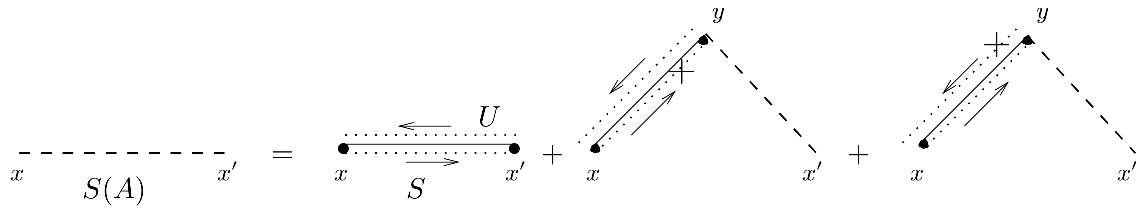

Figure 1: Graphical representation of Eq. (3.12). The double line (one full and one dotted joining two circles) represents the gauge invariant Green's function $S_{(1)} \equiv S$ [Eq. (2.18)] with a path along a single straight line; the single dotted line represents the phase factor, the dashed line the quark propagator in the external gluon field, the arrow the orientation on the path. The cross represents the rigid path derivation with one end fixed; it is placed near the end point which is submitted to derivation. y is an integration variable.

4 Functional relations for Green's functions

Functional relations between various gauge invariant quark Green's functions are obtained with a systematic use of Eqs. (3.11) or (3.12).

Let us consider the Green's function $S_{(n)}$ [Eq. (2.17)]. Integrating with respect to the quark fields, one obtains:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = \frac{1}{N_c}\,\langle U(x',y_{n-1})U(y_{n-1},y_{n-2})\cdots U(y_1,x)S(x,x';A)\rangle. \qquad (4.1)$$

The simplest case of this equation, corresponding to $n=1$, is:

$$S_{(1)}(x,x') \equiv S(x,x') = \frac{1}{N_c}\,\langle U(x',x)S(x,x';A)\rangle. \qquad (4.2)$$

The quark field integration yields also a corresponding determinant, which is a functional of the quark propagator in the external gluon field A. That determinant will not, however, play an active role in the subsequent calculations and hence will not explicitly appear in the various formulas that we shall meet; it will rather contribute as a background effect; in particular, it contributes to the evaluation of the Wilson loop averages, unless the quenched approximation is adopted. Therefore, the averaging formulas that we shall encounter should be understood with the presence of the quark field determinant. The expansions that will be used for the quark propagator in the external gluon field can also be repeated inside the quark field determinant if Wilson loop averages are to be evaluated.

Using now for $S(A)$ Eq. (3.12), one obtains:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = \frac{1}{N_c}S(x,x')\langle U(x',y_{n-1})\cdots U(y_1,x)U(x,x')\rangle$$
$$+\frac{1}{N_c}\langle U(x',y_{n-1})\cdots U(y_1,x)\Big(\frac{\bar{\delta}S(x,y_n)}{\bar{\delta}y_n^{\alpha+}}U(x,y_n) + S(x,y_n)\frac{\bar{\delta}U(x,y_n)}{\bar{\delta}y_n^{\alpha-}}\Big)\gamma^\alpha S(y_n,x';A)\rangle$$

$$= S(x,x')\, e^{F_{n+1}(x', y_{n-1}, \ldots, y_1, x)} \\ +\Big(\frac{\bar{\delta} S(x,y_n)}{\bar{\delta} y_n^{\alpha +}} + S(x,y_n)\frac{\bar{\delta}}{\bar{\delta} y_n^{\alpha -}}\Big)\, \gamma^\alpha\, S_{(n+1)}(y_n, x'; y_{n-1}, \ldots, y_1, x). \tag{4.3}$$

A graphical representation of this equation for $n = 3$ is shown in Fig. 2.

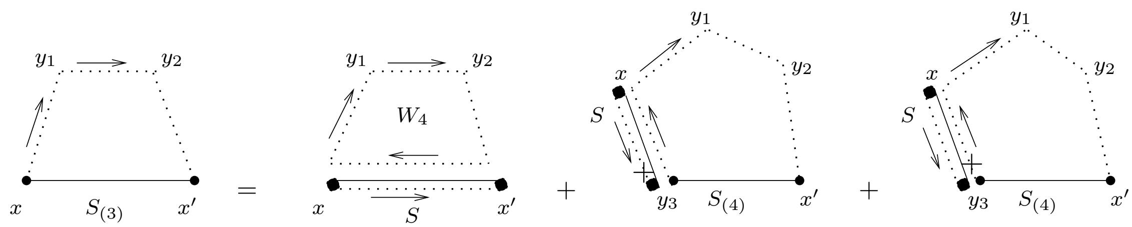

Figure 2: Graphical representation of Eq. (4.3) for $n = 3$. Same conventions as in Fig. 1. y_3 is an integration variable.

The Green's function $S_{(n)}$ satisfies the following equation of motion with x:

$$(i\gamma.\partial_{(x)} - m)S_{(n)}(x,x'; y_{n-1}, \ldots, y_1) = i\delta^4(x-x')e^{F_n(x, y_{n-1}, \ldots, y_1)} \\ +i\gamma^\mu \frac{\bar{\delta} S_{(n)}(x,x'; y_{n-1}, \ldots, y_1)}{\bar{\delta} x^{\mu -}}. \tag{4.4}$$

A graphical representation of this equation for $n = 1$ and $n = 3$ is shown in Fig. 3.

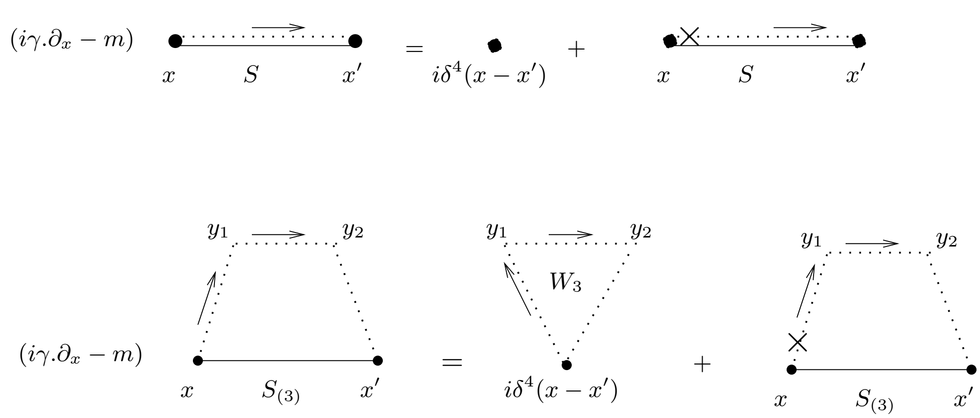

Figure 3: Graphical representation of the equations of motion of $S_{(1)} \equiv S$ and $S_{(3)}$. Same conventions as in Fig. 1.

5 Integral equation

The equations of motion of the gauge invariant Green's functions $S_{(n)}$ [Eqs. (3.8) and (4.4)] involve in their right-hand sides as unknowns the rigid path derivative of the Green's functions. The core of the problem amounts therefore to the evaluation of the rigid path derivative of Green's functions. That task, however, is facilitated by the functional relations (4.3), which relate two successive Green's functions with increasing index. They allow the evaluation of the rigid path derivative of a Green's function in terms of a similar derivative of a Wilson loop average and the derivative of a Green's function with a higher index. Systematic repetition of this procedure allows us therefore to express the rigid path derivative of a Green's function in terms of a series of Green's functions whose coefficients are functional derivatives of Wilson loop averages. One thus obtains chains of coupled integral (or integro-differential) equations between the various Green's functions. At the end, each Green's function $S_{(n)}$ can be expressed, at leading order of an expansion, by means of the functional relation (4.3), in terms of the lowest-order Green's function S, and thus an equation where solely the Green's function S would appear becomes reachable.

In the present work we are mainly interested by the simplest Green's function S and therefore we shall concentrate our considerations on the equation of motion of that quantity.

The rigid path derivative of $S_{(n)}$ along the segment xy_1 is obtained from Eq. (4.3):

$$\frac{\bar{\delta}S_{(n)}(x,x';y_{n-1},\ldots,y_1)}{\bar{\delta}x^{\mu-}} = \frac{\bar{\delta}F_{n+1}}{\bar{\delta}x^{\mu-}}\,e^{F_{n+1}(x',y_{n-1},\ldots,y_1,x)}\,S(x,x') \\ +\frac{\bar{\delta}}{\bar{\delta}x^{\mu-}}\Big(\frac{\bar{\delta}S(x,y_n)}{\bar{\delta}y_n^{\alpha+}} + S(x,y_n)\frac{\bar{\delta}}{\bar{\delta}y_n^{\alpha-}}\Big)\,\gamma^{\alpha}\,S_{(n+1)}(y_n,x';y_{n-1},\ldots,y_1,x). \tag{5.1}$$

Eliminating in the right-hand side of the latter equation the product $e^{F_{n+1}}S$ through Eq. (4.3), one obtains the equation

$$\frac{\bar{\delta}S_{(n)}(x,x';y_{n-1},\ldots,y_1)}{\bar{\delta}x^{\mu-}} = \frac{\bar{\delta}F_{n+1}(x',y_{n-1},\ldots,y_1,x)}{\bar{\delta}x^{\mu-}}\,S_{(n)}(x,x';y_{n-1},\ldots,y_1) \\ +\Big(\frac{\bar{\delta}}{\bar{\delta}x^{\mu-}} - \frac{\bar{\delta}F_{n+1}}{\bar{\delta}x^{\mu-}}\Big)\Big(\frac{\bar{\delta}S(x,y_n)}{\bar{\delta}y_n^{\alpha+}} + S(x,y_n)\frac{\bar{\delta}}{\bar{\delta}y_n^{\alpha-}}\Big)\,\gamma^{\alpha}\,S_{(n+1)}(y_n,x';y_{n-1},\ldots,y_1,x). \tag{5.2}$$

[Integrations on new variables in the right-hand sides are implicit.] For $n=1$, one has:

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} = \frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}\,S(x,x') \\ +\Big(\frac{\bar{\delta}}{\bar{\delta}x^{\mu-}} - \frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}\Big)\Big(\frac{\bar{\delta}S(x,y_1)}{\bar{\delta}y_1^{\alpha_1+}} + S(x,y_1)\frac{\bar{\delta}}{\bar{\delta}y_1^{\alpha_1-}}\Big)\,\gamma^{\alpha_1}\,S_{(2)}(y_1,x';x). \tag{5.3}$$

We next evaluate, in Eq. (5.3), the action of the path derivation operators on $S_{(2)}$. The operator $\bar{\delta}/\bar{\delta}x^{\mu-}$ acts here on the segment xx' and therefore it can be brought without

harm to the utmost right, where an equation similar to Eq. (5.2) (with a relabelling of some arguments, the point x being now a junction point on the path y_1xx' of $S_{(2)}$) is used with $n=2$ and then $\bar{\delta}F_3/\bar{\delta}x^{\mu-}$ is brought back to the left; during the last operation it is also submitted to the action of the operator $\bar{\delta}/\bar{\delta}y^{\alpha_1-}$. The resulting terms that involve $S_{(2)}$ are:

$$\Big(\frac{\bar{\delta}F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}}-\frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}\Big)\Big(\frac{\bar{\delta}S(x,y_1)}{\bar{\delta}y_1^{\alpha_1+}}+S(x,y_1)\frac{\bar{\delta}}{\bar{\delta}y_1^{\alpha_1-}}\Big)\,\gamma^{\alpha_1}\,S_{(2)}(y_1,x';x)$$
$$+\frac{\bar{\delta}^2F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1-}}\,S(x,y_1)\,\gamma^{\alpha_1}\,S_{(2)}(y_1,x';x). \tag{5.4}$$

Next, one observes that $\bar{\delta}S_{(2)}/\bar{\delta}y^{\alpha_1-}$ is part of the equation of motion of $S_{(2)}$ [Eq. (4.4)], y_1 being now one of the fermionic ends of $S_{(2)}$. Using the latter equation and making an integration by parts with respect to y_1, one arrives at a simplified expression in which in the second derivative of F_3 the derivation $\bar{\delta}/\bar{\delta}y_1^{\alpha_1-}$, which is along the segment y_1x, is replaced by the derivation $\bar{\delta}/\bar{\delta}y_1^{\alpha_1+}$, which is along y_1x'. [The delta functions of the equations of motion do not contribute here because of the existence of the difference term $(\bar{\delta}F_3/\bar{\delta}x^{\mu-}-\bar{\delta}F_2/\bar{\delta}x^{\mu-})$.]

The net result is, including also the resulting $S_{(3)}$ terms:

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}}=\frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}\,S(x,x')-\frac{\bar{\delta}^2F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}}\,S(x,y_1)\,\gamma^{\alpha_1}\,S_{(2)}(y_1,x';x)$$
$$+\Big(\frac{\bar{\delta}S(x,y_1)}{\bar{\delta}y_1^{\alpha_1+}}+S(x,y_1)\frac{\bar{\delta}}{\bar{\delta}y_1^{\alpha_1-}}\Big)\,\gamma^{\alpha_1}\,\Big(\frac{\bar{\delta}}{\bar{\delta}x^{\mu-}}-\frac{\bar{\delta}F_2(x,x')}{\bar{\delta}x^{\mu-}}\Big)$$
$$\times\Big(\frac{\bar{\delta}S(y_1,y_2)}{\bar{\delta}y_2^{\alpha_2+}}+S(y_1,y_2)\frac{\bar{\delta}}{\bar{\delta}y_2^{\alpha_2-}}\Big)\,\gamma^{\alpha_2}\,S_{(3)}(y_2,x';x,y_1). \tag{5.5}$$

In the term containing $S_{(3)}$, the factors with the derivatives with respect to x and y_2 are treated in the same way as were those with x and y_1 with $S_{(2)}$; they yield at the end the factor $\bar{\delta}^2F_4/\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}$ plus a term with $S_{(4)}$ having a similar structure than the one with $S_{(3)}$ above. Repeated use of the procedure described with $S_{(2)}$, yields a series expansion in $S_{(n)}$ $(n=2,3,\ldots)$ where all terms have similar structures. One obtains:

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}}=\frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}\,S(x,x')-\frac{\bar{\delta}^2F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}}\,S(x,y_1)\,\gamma^{\alpha_1}\,S_{(2)}(y_1,x';x)$$
$$-\Big(\frac{\bar{\delta}S(x,y_1)}{\bar{\delta}y_1^{\alpha_1+}}+S(x,y_1)\frac{\bar{\delta}}{\bar{\delta}y_1^{\alpha_1-}}\Big)\,\gamma^{\alpha_1}\,\frac{\bar{\delta}^2F_4(x',x,y_1,y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}}\,S(y_1,y_2)\,\gamma^{\alpha_2}\,S_{(3)}(y_2,x';x,y_1)$$
$$-\sum_{n=4}^{\infty}\Big(\frac{\bar{\delta}S(x,y_1)}{\bar{\delta}y_1^{\alpha_1+}}+S(x,y_1)\frac{\bar{\delta}}{\bar{\delta}y_1^{\alpha_1-}}\Big)\,\gamma^{\alpha_1}$$
$$\times\cdots\times\ \Big(\frac{\bar{\delta}S(y_{n-3},y_{n-2})}{\bar{\delta}y_{n-2}^{\alpha_{n-2}+}}+S(y_{n-3},y_{n-2})\frac{\bar{\delta}}{\bar{\delta}y_{n-2}^{\alpha_{n-2}-}}\Big)\gamma^{\alpha_{n-2}}$$
$$\times\,\frac{\bar{\delta}^2F_{n+1}(x',x,y_1,\ldots,y_{n-1})}{\bar{\delta}x^{\mu-}\bar{\delta}y_{n-1}^{\alpha_{n-1}+}}\,S(y_{n-2},y_{n-1})\,\gamma^{\alpha_{n-1}}\,S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2}). \tag{5.6}$$

Equation (5.6) determines the action of the rigid path derivative on the Green's function. It is expressed in terms of derivatives of logarithms of Wilson loop averages appearing in a series of terms with skew-polygonal type contours. Although the resulting expression still contains other derivatives along internal lines, as well as implicit terms, their action will be determined with respect to the structure defined by Eq. (5.6). The derivatives of the logarithms of Wilson loop averages, together with the accompanying quark Green's function S, play here the role of kernels of the integral equation we are searching for; they are the analogs of the kernels made of propagators in Feynman diagrams appearing in Dyson-Schwinger equations. By analogy, we shall often call them diagrams.

We now study the structure of the kernels that are present in the expansion (5.6). We notice that in the term with $S_{(n)}$, the utmost left derivative related to x is connected to the utmost right derivative related to y_{n-1} through the term $\bar{\delta}^2 F_{n+1}/\bar{\delta}x^-\bar{\delta}y_{n-1}^+$; this does not leave room for the existence of reducible type terms made of disjoint subsets of connnections; such terms are actually parts of the definitions of the $S_{(n)}$s when expanded in terms of free propagators. In the present case, all remaining derivatives $\bar{\delta}/\bar{\delta}y_i$ ($i = 1, \ldots, y_{n-2}$) either will act within the skew-polygonal line $xy_1\ldots y_{n-1}$ or will leave that line to be connected to a larger contour associated with an $S_{(m)}$ with $m > n$.

To exhibit the latter feature, we consider in Eq. (5.6) the general term $S_{(n)}$ and its associated derivative terms. We notice that the derivatives $\bar{\delta}/\bar{\delta}y_i$ ($i = 1, \ldots, y_{n-2}$) no longer act on the fermionic end of $S_{(n)}$ (represented by y_{n-1}) but only on the internal junction points of the skew-polygonal line $xy_1\ldots y_{n-1}$; hence, one cannot use the equation of motion of $S_{(n)}$ with these variables. We first consider the action of $\bar{\delta}/\bar{\delta}y_{n-2}^-$. It is brought to the right to act on $S_{(n)}$; during this operation it may also act on the term $\bar{\delta}^2 F_{n+1}/\bar{\delta}x^-\bar{\delta}y_{n-1}^+$ to yield a third-order derivative of F_{n+1}. When in front of $S_{(n)}$, it is convenient, to maintain symmetry with $\bar{\delta}/\bar{\delta}y_{n-1}^+$, to replace $\bar{\delta}/\bar{\delta}y_{n-2}^-$ by $(\partial/\partial y_{n-2} - \bar{\delta}/\bar{\delta}y_{n-2}^+)$; the total derivative is then used for an integration by parts to convert $\bar{\delta}^3 F_{n+1}/\bar{\delta}x^-\bar{\delta}y_{n-1}^+\bar{\delta}y_{n-2}^-$ into $\bar{\delta}^3 F_{n+1}/\bar{\delta}x^-\bar{\delta}y_{n-1}^+\bar{\delta}y_{n-2}^+$. The derivative $\bar{\delta}/\bar{\delta}y_{n-2}^+$ acting on $S_{(n)}$ has two kinds of effect, according to Eq. (4.3) (with a relabelling of variables): in the first place, it yields the derivative term $\bar{\delta}F_{n+1}/\bar{\delta}y_{n-2}^+$ and in the second, it generates a new Green's function $S_{(n+1)}$ with the fermionic end point y_n (the global result is very similar to that of Eq. (5.2) with a relabelling of variables). The term $\bar{\delta}F_{n+1}/\bar{\delta}y_{n-2}^+$ is an insertion along the line $xy_1\ldots y_{n-1}$ and may also be submitted to other derivations coming from the remaining variables y_j ($j = 1, \ldots, n-3$). In the newly generated term with $S_{(n+1)}$, the actions of the derivatives $\bar{\delta}/\bar{\delta}y_{n-2}^+$ and $\bar{\delta}/\bar{\delta}y_n^-$ can be combined to give $\bar{\delta}^2 F_{n+2}/\bar{\delta}y_{n-2}^+\bar{\delta}y_n^+$. This term represents now a connection between the line $xy_1\ldots y_{n-1}$ and the segment $y_{n-1}y_n$; it crosses the connection line xy_{n-1} which had already appeared with the term $\bar{\delta}^2 F_{n+1}/\bar{\delta}x^-\bar{\delta}y_{n-1}^+$; the two connections can therefore be interpreted as forming a crossed diagram. We observe that the appearance of a crossed diagram has been accompanied with the increase of the number of segments of the contour by one unit.

The action of the remaining derivatives $\bar{\delta}/\bar{\delta}y_j^-$ $(j=1,\ldots,n-3)$ can be studied in a similar way. The qualitative features displayed up to now remain unchanged.

We can summarize the above results by grouping the terms that appear in front of a Green's function $S_{(n)}$ into three categories: (i) A term that is completely connected: F_{n+1} is submitted to the n derivations $\bar{\delta}^n/\bar{\delta}x^-\bar{\delta}y_1^+\ldots\bar{\delta}y_{n-1}^+$. (ii) Crossed diagrams that involve at least one F_{n+1} and some other Fs with lower indices. (iii) Nested diagrams, represented by insertions within the connection line $xy_1\ldots y_{n-1}$ or within smaller connections of that line or within crossed diagrams. As a general property, no terms of reducible type (disjoint connections) exist.

The general structure of the derivative $\bar{\delta}S/\bar{\delta}x^{\mu-}$ is:

$$\begin{aligned}\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} &= K_{1\mu-}(x',x)\,S(x,x')+K_{2\mu-}(x',x,y_1)\,S_{(2)}(y_1,x';x)\\ &\quad+\sum_{i=3}^{\infty}K_{i\mu-}(x',x,y_1,\ldots,y_{i-1})\,S_{(i)}(y_{i-1},x';x,y_1,\ldots,y_{i-2}),\end{aligned}\tag{5.7}$$

where the kernels K_i $(i=1,\ldots,\infty)$ are composed of the three categories of terms quoted above and of $(i-1)$ quark propagators S, and eventually of their derivatives, along the segments of the $(i+1)$-sided skew-polygons. The total number of derivatives contained in K_n is n.

The explicit expression of $\bar{\delta}S/\bar{\delta}x^{\mu-}$ up to the fourth-order of its expansion is:

$$\begin{aligned}\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} &= \frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}\,S(x,x')-\frac{\bar{\delta}^2F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}}\,S(x,y_1)\,\gamma^{\alpha_1}\,S_{(2)}(y_1,x';x)\\
&+\frac{\bar{\delta}^3F_4(x',x,y_1,y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}\bar{\delta}y_2^{\alpha_2+}}\,S(x,y_1)\,\gamma^{\alpha_1}\,S(y_1,y_2)\,\gamma^{\alpha_2}\,S_{(3)}(y_2,x';x,y_1)\\
&+\frac{\bar{\delta}^2F_4}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}}\,S(x,y_1)\,\gamma^{\alpha_1}\,\Big(\frac{\bar{\delta}S(y_1,y_2)}{\bar{\delta}y_1^{\alpha_1-}}+S(y_1,y_2)\frac{\bar{\delta}F_4}{\bar{\delta}y_1^{\alpha_1+}}\Big)\,\gamma^{\alpha_2}\,S_{(3)}(y_2,x';x,y_1)\\
&-\Big[\frac{\bar{\delta}^4F_5(x',x,y_1,y_2,y_3)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}\bar{\delta}y_2^{\alpha_2+}\bar{\delta}y_3^{\alpha_3+}}+\frac{\bar{\delta}^2F_4(x',x,y_1,y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}}\,\frac{\bar{\delta}^2F_5(x',x,y_1,y_2,y_3)}{\bar{\delta}y_1^{\alpha_1+}\bar{\delta}y_3^{\alpha_3+}}\Big]\\
&\times S(x,y_1)\,\gamma^{\alpha_1}\,S(y_1,y_2)\,\gamma^{\alpha_2}\,S(y_2,y_3)\,\gamma^{\alpha_3}\,S_{(4)}(y_3,x';x,y_1,y_2)\\
&-S(x,y_1)\,\gamma^{\alpha_1}\Big[\frac{\bar{\delta}^3F_5}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}\bar{\delta}y_3^{\alpha_3+}}\Big(\frac{\bar{\delta}S(y_1,y_2)}{\bar{\delta}y_1^{\alpha_1-}}+S(y_1,y_2)\frac{\bar{\delta}F_5}{\bar{\delta}y_1^{\alpha_1+}}\Big)\,\gamma^{\alpha_2}\,S(y_2,y_3)\\
&+\frac{\bar{\delta}^3F_5}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}\bar{\delta}y_3^{\alpha_3+}}\,S(y_1,y_2)\,\gamma^{\alpha_2}\,\Big(\frac{\bar{\delta}S(y_2,y_3)}{\bar{\delta}y_2^{\alpha_2-}}+S(y_2,y_3)\frac{\bar{\delta}F_5}{\bar{\delta}y_2^{\alpha_2+}}\Big)\\
&+\frac{\bar{\delta}^2F_5}{\bar{\delta}x^{\mu-}\bar{\delta}y_3^{\alpha_3+}}\Big(\frac{\bar{\delta}S(y_1,y_2)}{\bar{\delta}y_1^{\alpha_1-}}+S(y_1,y_2)\frac{\bar{\delta}F_5}{\bar{\delta}y_1^{\alpha_1+}}\Big)\,\gamma^{\alpha_2}\,\Big(\frac{\bar{\delta}S(y_2,y_3)}{\bar{\delta}y_2^{\alpha_2-}}+S(y_2,y_3)\frac{\bar{\delta}F_5}{\bar{\delta}y_2^{\alpha_2+}}\Big)\\
&+\frac{\bar{\delta}^2F_5}{\bar{\delta}x^{\mu-}\bar{\delta}y_3^{\alpha_3+}}\,\frac{\bar{\delta}^2F_5}{\bar{\delta}y_1^{\alpha_1+}\bar{\delta}y_2^{\alpha_2+}}\,S(y_1,y_2,)\,\gamma^{\alpha_2}\,S(y_2,y_3)\Big]\,\gamma^{\alpha_3}\,S_{(4)}(y_3,x';x,y_1,y_2)\\
&+\cdots\ .\end{aligned}\tag{5.8}$$

The expansion, up to third-order terms (including $S_{(3)}$), is represented graphically in Fig. 4.

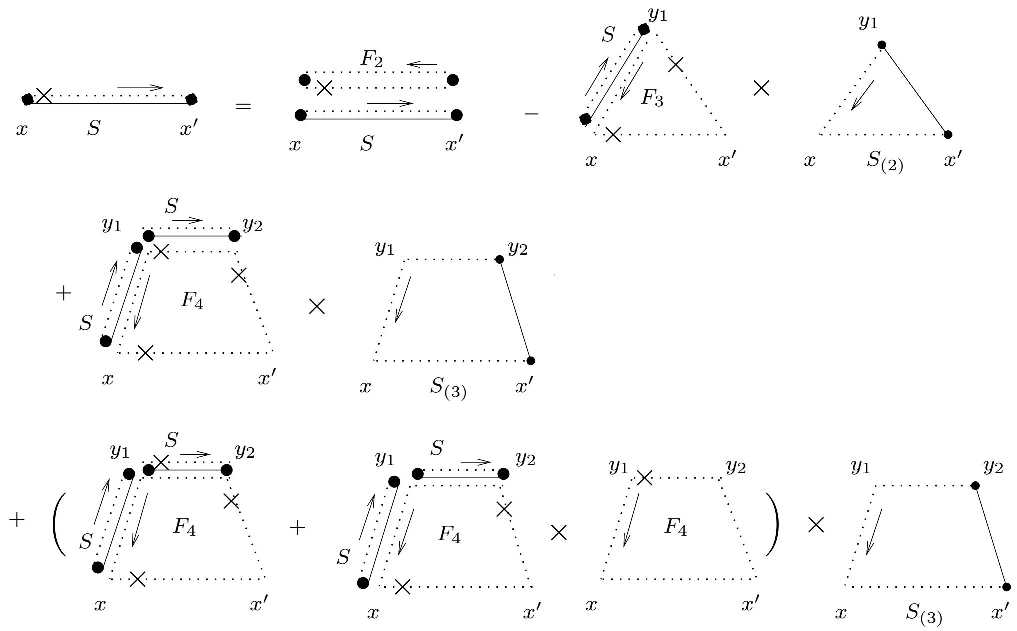

Figure 4: The expansion of $\bar{\delta}S/\bar{\delta}x^-$ up to third-order terms. Same conventions as in Fig. 1. The ys are integration variables.

In Eq. (5.8), the term in front of $S_{(2)}$ and the first terms in front of $S_{(3)}$ and $S_{(4)}$ correspond to the completely connected diagrams. The second term (within the first brackets) in front of $S_{(4)}$ represents a crossed diagram. The remaining terms with $S_{(3)}$ and $S_{(4)}$ correspond to nested diagrams.

The persistence in the integral equation of nested diagrams may seem puzzling. In the Dyson–Schwinger equation for the self-energy, once the internal propagators of diagrams are replaced by full propagators, no nested diagram survives; this is equivalent to stating that the summation of all internal diagrams in nested diagrams yields the full propagators. In the present formalism, the survival of nested diagrams is a consequence of a background effect induced by the Wilson loop; each sub-diagram is actually calculated in the presence of the Wilson loop appearing at the order of the global diagram. At each order of the expansion the contour of the relevant Wilson loop changes with an increase of the number of segments forming the contour. The insertion of a low-order diagram in a higher-order diagram thus modifies the expression of the former, since it is now expressed with the new Wilson loop derivatives. In dealing with ordinary Feynman diagrams, one does not encounter the above background generated effects.

These statements can be explicitly checked in Eq. (5.8). A way of ignoring background

effects can proceed as follows: (i) Assimilate an $n^{\rm th}$-order derivative of a function F_m ($m \geq n+1$) to an ordianary n-point function with propagators attached to the end points of the corresponding segments, with the convention that a derivation of the type $\bar{\delta}/\bar{\delta}y^-$ can be converted into a derivation of the type $\bar{\delta}/\bar{\delta}y^+$ with a change of sign. (ii) Consider first the approximation where $S_{(n)}$ is given by the first term of the expansion (4.3). (iii) When two loop contours corresponding to W_i and W_j have a common segment, replace the whole by a single contour corresponding now to W_{i+j-2}. It can then be verified that a systematic expansion of the terms $\bar{\delta}S/\bar{\delta}y^-$ inside the nested diagrams with an iterative procedure cancels order-by-order all other terms present in the nested diagrams and thus the latter disappear, as expected, when no background effects are retained. At a second stage, considering the further terms of the expression of $S_{(n)}$ [Eq. (4.3)] one finds that the latter are themselves background generated effects and the repetition of the above procedure makes them in turn disappear.

The rules of appearance of nested diagrams can be deduced from Eq. (5.6) and verified in Eq. (5.8).

In two dimensions, where the Wilson loop averages are determined by the areas of the surfaces lying inside the contours [13], the second functional derivatives of the functions F are delta-functions and in general the nested diagrams disappear. One might expect that, in four dimensions, the nested diagrams, even if not disappearing, remain negligible on quantitative grounds. A more complete idea of their role could be obtained only when renormalization properties of the integral equation are studied.

Equation (5.6), together with relations (5.2), allows the calculation of the term $\bar{\delta}S/\bar{\delta}x^-$ through an expansion involving an increasing number of functional derivatives of Wilson loops. The calculation of the expression of the kernel K_n appearing in the expansion (5.7) requires solely consideration of terms of order lower or equal to n. The integral form of the equation of motion (3.8) is:

$$S(x,x') = S_0(x,x') + \int d^4x''\, S_0(x,x'')\, \gamma^\mu\, \frac{\bar{\delta}S(x'',x')}{\bar{\delta}x''^{\mu -}}, \tag{5.9}$$

in which one has to inject the expression of $\bar{\delta}S/\bar{\delta}x^-$ resulting from Eq. (5.7).

At short-distances, governed by perturbation theory, each derivation introduces a new power of the coupling constant and therefore the dominant terms in the expansion are the lowest-order ones. At large-distances, Wilson loops are saturated by the minimal surfaces having as supports the contours [7, 8, 26]. Here also, the dominant contributions come from the lowest-order derivative terms. Therefore the expansion in Eq. (5.7) can be considered in general as a perturbative one whatever the distances are, provided that for each type of region the appropriate expressions are used for the Wilson loops. The first term of the expansion, represented by a single derivative, is null for symmetry reasons (the derivative $\bar{\delta}F_2(x',x)/\bar{\delta}x^-$ being orthogonal to xx'). Hence the non-zero dominant term of the expansion is the second-order derivative term. Furthermore, the various Green's

functions $S_{(n)}$ are themselves dominated by their lowest-order expression of Eq. (4.3), involving only S and a Wilson loop. In that approximation, $\bar{\delta}S(x,x')/\bar{\delta}x^-$ takes the form

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu -}} \simeq -\int d^4y_1\, \frac{\bar{\delta}^2 F_3(x',x,y_1)}{\bar{\delta}x^{\mu -}\bar{\delta}y_1^{\alpha_1 +}}\, e^{F_3(x',x,y_1)}\, S(x,y_1)\,\gamma^{\alpha_1}\, S(y_1,x'). \tag{5.10}$$

Thus, the dominant part of the integral equation (5.9) to be solved reduces to a closed form involving only the full propagator S, the free propagator S_0, a Wilson loop average and its second-order rigid path derivative.

The lack of manifest symmetry in Eq. (5.9) between the coordinates x and x' is due to the presence of the closed contours of the Wilson loops, which do not allow immediate factorization of propagators through convolution operations. Nevertheless, because of translation invariance, the Green's function $S(x,x')$ depends only on the difference $(x-x')$; therefore, once the integrations in Eq. (5.9) are done, one should recover the desired symmetry. If, instead of the equation of motion (3.8), relative to x, we had used the equation of motion (3.9), relative to x', we would have found an integral equation where $S_0(x'',x')$ acts on $\bar{\delta}S(x,x'')/\bar{\delta}x''^{\mu+}\gamma^\mu$ from the right.

As a complementary remark with respect to the method of approach developed in this work, we point out that another way of proceeding would consist of trying to sum diagrams constructed with free quark propagators associated with phase factors. This method actually corresponds to the use of the expansion equations (3.4) and (3.5), instead of (3.11) and (3.12). It, however, becomes rapidly intricate due to the presence of the background Wilson loop effects. Nevertheless, the first few terms of Eqs. (5.7) and (5.8) can be reconstituted rather easily. The main aspects of this method are presented in Appendix A.

The integral equation (5.9), together with the expressions (5.7) or (5.8), did not make any explicit reference to the quark self-energy function. The latter can be obtained once the Green's function S is calculated, through its inverse. It is, however, also possible to construct it directly, by setting up a specific equation for it. This is presented in Appendix B.

The problem of solving Eq. (3.8), or its equivalent Eq. (5.9), together with expression (5.7), leads us to the question of representation of the gauge invariant two-point Green's function. This will be considered in Sec. 6.

6 Analyticity properties of the Green's function

One of the advantages of paths along straight lines is the fact that the expressions of the corresponding Green's functions become dependent only on the end points of the paths. This feature in turn allows a simple transition to momentum space by Fourier transformation. Much of the informations on Green's functions are provided from momentum space, since it is there that their spectral properties are determined.

From this point of view, the quark two-point gauge invariant Green's functions hold a particular position. Because of confinement of colored objects, it is not possible to cut the path joining the quark to the antiquark by inserting in it a complete set of physical states, which are color singlets. This feature seems to suggest that gauge invariant two-point Green's functions should not have any singularities.

The situation is, however, more complex than it seems. Gauge invariant two-point Green's functions possess singularities originated from perturbation theory. This is corroborated by the integral equation (5.9), in which the presence of the free quark propagator generates new singularities in the complete solution. An analysis, starting from perturbation theory, is therefore necessary.

We shall admit that, in a domain where perturbation theory is valid, it is meaningful to consider quarks and gluons as physical particles with positive energies, described by corresponding physical states. It is then advantageous to consider the path-ordered phase factor (2.1) in its representation given by the series expansion in terms of the coupling constant g, the $n^{\rm th}$-order term of the expansion containing $(n-1)$ gluon fields $(n \geq 1)$.

Adopting here an operator formalism, we observe that the gauge invariant quark two-point function involves two kinds of orderings for its defining fields. The first is the path-ordering (or P-ordering) which concerns the color index arrangements of the gluon fields according to their positions on the path. The second is the time-ordering (T-ordering) or chronological product which enters in the definitions of Green's functions and operates once the P-ordering is done.

Another advantage of paths along straight lines is that once the timelike or spacelike nature of the distance between the quark and the antiquark is fixed, the nature of the mutual distances of the gluon fields in the Green's function S [Eq. (2.18)] is also fixed in the same way, because of their alignement along the segment joining the quark to the antiquark. Therefore, the chronological product of the $n^{\rm th}$-order terms in S reduces to two terms, defined by the relative time between the quark and the antiquark. According to the definitions (2.1) and (2.18), for timelike $(x'-x)$, if $(x'^0-x^0) > 0$ the T-ordering will coincide with the P-ordering, while if $(x'^0-x^0) < 0$ the T-ordering will be the opposite of the P-ordering (with a change of sign for the fermion fields), the color indices being already fixed from the P-ordering. We are in a situation which is very similar to the case of the ordinary two-point function, with the difference that for an $n^{\rm th}$-order term there are $(n+1)$ fields instead of two ($(n-1)$ gluon, one quark and one antiquark fields).

Using for each of the two products which make the T-product the spectral analysis with intermediate states, taking into account the bounds on the parameters of the P-ordering and using causality, one arrives at a generalized form of the Källén–Lehmann representation for the Green's function S in momentum space, in which the cut starts on the real axis from the quark mass squared m^2 and extends to infinity [27, 28, 29, 30, 31]. The generalization is due to the fact that each gluon field is integrated along the path and

this introduces, when using for the latter a dimensionless parameter λ varying between 0 and 1, a multiplicative factor $(x' - x)$, which is converted in momentum space into a derivation operator; each such factor increases by one unit the power of the denominator of the dispersion integral. Finally, because of the fact that we are dealing here with a gauge invariant quantity, we expect not to encounter at the end spurious infrared divergences.

To summarize the above results, we introduce the Fourier transform of the Green's function $S(x, x')$, for which we also take into account translation invariance:

$$S(x, x') = S(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip.(x - x')} S(p). \tag{6.1}$$

$S(p)$ has the following representation in terms of real spectral functions $\rho_1^{(n)}$ and $\rho_0^{(n)}$ $(n = 1, \ldots, \infty)$:

$$S(p) = i \int_0^\infty ds' \sum_{n=1}^{\infty} \frac{[\, \gamma.p\, \rho_1^{(n)}(s') + \rho_0^{(n)}(s')\,]}{(p^2 - s' + i\varepsilon)^n}. \tag{6.2}$$

This is a conservative representation of the various contributions encountered above; simplifications or recombinations into more compact forms might still occur. It is evident that formally the powers of the denominators can be lowered by integration by parts; however, possible singularities of the spectral functions at threshold could prevent such an operation.

We assume that the above representation, obtained from the domain of perturbation theory, remains also valid in non-perturbative regimes. One expects that the resulting singularities are strong enough to screen the quark pole and other physical type singularities.

Further study is needed to define more accurately the properties of the spectral functions. Nevertheless, representation (6.2), or another one related to it, might be tried for the investigation of the solutions of the corresponding integral equation.

7 Summary and comments

We have expressed the equation of motion of the gauge invariant quark two-point function having a straight line path as an integral or integro-differential equation involving the series of all two-point functions with paths of skew-polygonal type, in which the kernels are given by quark Green's functions and rigid path derivatives of the logarithms of the Wilson loop averages with contours made of these lines.

Gauge invariant quark Green's functions also satisfy, in addition to their equations of motion related to the quark ends, other equations of motion resulting from local deformations of their paths. The latter equations are typically those of the path-ordered phase factors and once Wilson loops are introduced through the calculations, they reduce to the characteristic equations of Wilson loops, i.e., to the Bianchi identity and to the loop

equation or Makeenko–Migdal equation [6, 7, 8]. We have not insisted on that aspect of the problem, since it has been widely studied in the literature. This implies that when the Wilson loop averages are used in the kernels of the above integral equations, they are understood as being solutions (at least approximately) of their own equations of motion.

We have emphasized the fact that the series of kernels appearing in the integral equation can be considered, on quantitative grounds, as a perturbation series simultaneously for short- and large-distances and therefore could be approximated, for a starting calculation, by its lowest-order non-vanishing term.

A question which was not considered in the present work concerns the renormalization properties of the gauge invariant Green's functions. These seem to be intimately related, through the integral equation, to those of the Wilson loop averages [14, 15] and could be dealt with only when explicit expressions of the latter are introduced.

The method of functional relations between two-point Green's functions with different numbers of segments along their paths, can also be applied, with appropriate generalizations, to other n-point Green's functions, with $n \geq 4$. That problem is particularly relevant for the derivation of a bound state equation for a quark-antiquark system.

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A Summation method

Integral equations of ordinary Green's functions represent in general the result of summing the classes of reducible diagrams in terms of free propagators. One should expect that a similar procedure might also be operative in the case of gauge invariant Green's functions.

To implement this method of approach, we should start with expressions involving free quark propagators. To this end, we have to use either of the two representations (3.4) or (3.5) of the quark propagator in external field and expand $S(A)$ in the defining equation of S [Eq. (4.2)]. Each of the two representations has its own advantages and could be preferred for a definite aim. Thus, representation (3.4) is more appropriate to obtain rapidly the structure resulting from equations of motion with respect to x, while representation (3.5) is more appropriate for the calculation of $\bar{\delta}S(x,x')/\bar{\delta}x^-$.

Using first representation (3.4), the expansion of $S(A)$ in Eq. (4.2) generates Wilson loops with skew-polygonal type contours, accompanied with free quark propagators:

$$S(x,x') = S_0(x,x')\, e^{F_2(x',x)} - S_0(x,y_1)\, \gamma^{\alpha_1}\, S_0(y_1,x')\, \frac{\bar{\delta}}{\bar{\delta}y_1^{\alpha_1+}}\, e^{F_3(x',x,y_1)}$$
$$+ \sum_{j=2}^{\infty}(-1)^j\, S_0(x,y_1)\, \gamma^{\alpha_1}\, S_0(y_1,y_2)\, \gamma^{\alpha_2}\, \cdots\, \gamma^{\alpha_j}\, S_0(y_j,x')$$

$$\times\frac{\bar\delta}{\bar\delta y_1^{\alpha_1+}}\frac{\bar\delta}{\bar\delta y_2^{\alpha_2+}}\cdots\frac{\bar\delta}{\bar\delta y_j^{\alpha_j+}}\, e^{F_{j+2}(x',x,y_1,\ldots,y_j)}. \tag{A.1}$$

A similar expansion can also be done for $S_{(n)}$ $(n>1)$, starting from Eq. (4.1):

$$\begin{aligned}
S_{(n)}(x,x';z_{n-1},\ldots,z_1) &= S_0(x,x')\, e^{F_{n+1}(x',z_{n-1},\ldots,z_1,x)}\\
&-S_0(x,y_1)\,\gamma^{\alpha_1}\,S_0(y_1,x')\,\frac{\bar\delta}{\bar\delta y_1^{\alpha_1+}}\, e^{F_{n+2}(x',z_{n-1},\ldots,z_1,x,y_1)}\\
&+\sum_{j=2}^{\infty}(-1)^j\,S_0(x,y_1)\,\gamma^{\alpha_1}\,S_0(y_1,y_2)\,\gamma^{\alpha_2}\,\cdots\,\gamma^{\alpha_j}\,S_0(y_j,x')\\
&\quad\times\frac{\bar\delta}{\bar\delta y_1^{\alpha_1+}}\frac{\bar\delta}{\bar\delta y_2^{\alpha_2+}}\cdots\frac{\bar\delta}{\bar\delta y_j^{\alpha_j+}}\, e^{F_{n+j+1}(x',z_{n-1},\ldots,z_1,x,y_1,\ldots,y_j)}.
\end{aligned} \tag{A.2}$$

Use of representation (3.5) yields equivalent expressions for S and $S_{(n)}$ $(n>1)$:

$$\begin{aligned}
S(x,x') &= \sum_{j=0}^{\infty}\,S_0(x,y_1)\,\gamma^{\alpha_1}\,S_0(y_1,y_2)\,\gamma^{\alpha_2}\,\cdots\,\gamma^{\alpha_j}\,S_0(y^j,x')\\
&\quad\times\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\frac{\bar\delta}{\bar\delta y_2^{\alpha_2-}}\cdots\frac{\bar\delta}{\bar\delta y_j^{\alpha_j-}}\, e^{F_{j+2}(x',x,y_1,\ldots,y_j)},
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
S_{(n)}(x,x';z_{n-1},\ldots,z_1) &= \sum_{j=0}^{\infty}\,S_0(x,y_1)\,\gamma^{\alpha_1}\,S_0(y_1,y_2)\,\gamma^{\alpha_2}\,\cdots\,\gamma^{\alpha_j}\,S_0(y^j,x')\\
&\quad\times\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\frac{\bar\delta}{\bar\delta y_2^{\alpha_2-}}\cdots\frac{\bar\delta}{\bar\delta y_j^{\alpha_j-}}\, e^{F_{n+j+1}(x',z_{n-1},\ldots,z_1,x,y_1,\ldots,y_j)}.
\end{aligned} \tag{A.4}$$

Equations (A.3) and (A.4) could also have been obtained from Eqs. (A.1) and (A.2), respectively, by integrations by parts; at the internal junction points y_i of the segments of the paths, one has the equivalence relations $\partial/\partial y_i = \bar\delta/\bar\delta y_i^+ + \bar\delta/\bar\delta y_i^-$ [Eq. (2.10)].

The equations of motion with respect to x can be evaluated easily from representations (A.1) and (A.2). Because of the appearance of delta-functions from the propagators $S_0(x,y_1)$, there are cancellations between successive terms and one finds:

$$\begin{aligned}
(i\gamma.\partial_{(x)}-m)S(x,x') &= i\delta^4(x-x')\\
&+\sum_{j=0}^{\infty}(-1)^j\,i\gamma^\mu\,S_0(x,y_1)\,\gamma^{\alpha_1}\,S_0(y_1,y_2)\,\gamma^{\alpha_2}\,\cdots\,\gamma^{\alpha_j}\,S_0(y_j,x')\\
&\times\frac{\bar\delta}{\bar\delta x^{\mu-}}\frac{\bar\delta}{\bar\delta y_1^{\alpha_1+}}\frac{\bar\delta}{\bar\delta y_2^{\alpha_2+}}\cdots\frac{\bar\delta}{\bar\delta y_j^{\alpha_j+}}\, e^{F_{j+2}(x',x,y_1,\ldots,y_j)}\\
&= i\delta^4(x-x') + i\gamma^\mu\,\frac{\bar\delta S(x,x')}{\bar\delta x^{\mu-}},
\end{aligned} \tag{A.5}$$

$$
\begin{aligned}
&(i\gamma.\partial_{(x)} - m)S_{(n)}(x,x';z_{n-1},\ldots,z_1) = i\delta^4(x-x')\,e^{F_n(x,z_{n-1},\ldots,z_1)}\\
&\qquad + \sum_{j=0}^{\infty}(-1)^j\, i\gamma^\mu\, S_0(x,y_1)\,\gamma^{\alpha_1}\,S_0(y_1,y_2)\,\gamma^{\alpha_2}\,\cdots\,\gamma^{\alpha_j}\,S_0(y_j,x')\\
&\qquad \times \frac{\bar\delta}{\bar\delta x^{\mu-}}\frac{\bar\delta}{\bar\delta y_1^{\alpha_1+}}\frac{\bar\delta}{\bar\delta y_2^{\alpha_2+}}\cdots\frac{\bar\delta}{\bar\delta y_j^{\alpha_j+}}\,e^{F_{n+j+1}(x',z_{n-1},\ldots,z_1,x,y_1,\ldots,y_j)}\\
&\qquad = i\delta^4(x-x')\,e^{F_n(x,z_{n-1},\ldots,z_1)} + i\gamma^\mu\,\frac{\bar\delta S_{(n)}(x,x';z_{n-1},\ldots,z_1)}{\bar\delta x^{\mu-}}.
\end{aligned}
\tag{A.6}
$$

Considering representation (A.3), one immediately checks that it has the structure of the integral equation (5.9); this represents of course the integrated form of the equation of motion (A.5).

The action of the rigid path derivatives on the exponential functionals in Eqs. (A.1)-(A.4) can be evaluated easily. The aim is then to group the various terms that appear in the expression of $\bar\delta S(x,x')/\bar\delta x^-$ to bring the latter into the form of Eq. (5.8). We shall do this in a perturbative expansion with respect to the number of derivations, by retaining up to derivatives of third order and showing that Eq. (5.8) can be obtained up to the $S_{(3)}$ terms. That approximation is sufficient to illustrate the various aspects of the method under consideration.

For the calculation of $\bar\delta S(x,x')/\bar\delta x^-$, it is preferable to start with representations (A.3) and (A.4) of S and $S_{(n)}$; we shall indicate below the specific differences one meets when starting with representations (A.1) and (A.2), although the final result is the same. The advantage of the former representations is that when a derivation $\bar\delta/\bar\delta y_i^-$ acts on $S_{(i+1)}$ in which y_i is a fermionic end, it can be directly replaced in terms of the equation of motion operator and a delta-function, allowing an integration by parts. This is not the case with the operator $\bar\delta/\bar\delta y_i^+$, which first should be transformed into $\bar\delta/\bar\delta y_i^-$ before an integration by parts be possible. In the expansions (A.3) and (A.4), S will be approximated with the first three terms, $S_{(2)}$ with the first two and $S_{(3)}$ with the first term. We shall also assume the backtracking property of the path-ordered phase factors [10], which means that $U(y,x)U(x,y) = 1$, the same path being run forth and back. This means that $W_2 = 1$ and $F_2 = 0$.

Calculating $\bar\delta S(x,x')/\bar\delta x^-$ from Eq. (A.3) one obtains:

$$
\begin{aligned}
\frac{\bar\delta S(x,x')}{\bar\delta x^{\mu-}} &= S_0(x,x')\,\frac{\bar\delta}{\bar\delta x^{\mu-}}e^{F_2(x',x)} + S_0(x,y_1)\gamma^{\alpha_1}S_0(y_1,x')\,\frac{\bar\delta}{\bar\delta x^{\mu-}}\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\,e^{F_3(x',x,y_1)}\\
&+S_0(x,y_1)\,\gamma^{\alpha_1}\,S_0(y_1,y_2)\,\gamma^{\alpha_2}\,S_0(y_2,x')\,\frac{\bar\delta}{\bar\delta x^{\mu-}}\frac{\bar\delta}{\bar\delta y_1^{\alpha_1-}}\frac{\bar\delta}{\bar\delta y_2^{\alpha_2-}}\,e^{F_4(x',x,y_1,y_2)}\\
&+\cdots\ .
\end{aligned}
\tag{A.7}
$$

Calculating the derivative $\bar\delta/\bar\delta x^-$, bringing the result to the left and completing in the first, second and third terms of the right-hand side of Eq. (A.7) the functions S, $S_{(2)}$ and

$S_{(3)}$, respectively, we obtain:

$$\frac{\bar{\delta} S(x,x')}{\bar{\delta} x^{\mu -}} = \frac{\bar{\delta} F_2(x',x)}{\bar{\delta} x^{\mu -}}\, S(x,x') + S_0(x,y_1)\gamma^{\alpha_1}\Big[\Big(\frac{\bar{\delta} F_3(x',x,y_1)}{\bar{\delta} x^{\mu -}} - \frac{\bar{\delta} F_2(x',x)}{\bar{\delta} x^{\mu -}}\Big)\frac{\bar{\delta}}{\bar{\delta} y_1^{\alpha_1 -}}$$
$$+\frac{\bar{\delta}^2 F_3(x',x,y_1)}{\bar{\delta} x^{\mu -}\bar{\delta} y_1^{\alpha_1 -}}\Big] S_{(2)}(y_1,x';x)$$
$$+S_0(x,y_1)\gamma^{\alpha_1} S_0(y_1,y_2)\gamma^{\alpha_2}\Big[\Big(\frac{\bar{\delta} F_4(x',x,y_1,y_2)}{\bar{\delta} x^{\mu -}} - \frac{\bar{\delta} F_3(x',x,y_1)}{\bar{\delta} x^{\mu -}}\Big)\frac{\bar{\delta}}{\bar{\delta} y_1^{\alpha_1 -}}\frac{\bar{\delta}}{\bar{\delta} y_1^{\alpha_2 -}}$$
$$+\Big(\frac{\bar{\delta}^2 F_4}{\bar{\delta} x^{\mu -}\bar{\delta} y_1^{\alpha_1 -}} - \frac{\bar{\delta}^2 F_3}{\bar{\delta} x^{\mu -}\bar{\delta} y_1^{\alpha_1 -}}\Big)\frac{\bar{\delta}}{\bar{\delta} y_2^{\alpha_2 -}}$$
$$+\frac{\bar{\delta}^2 F_4}{\bar{\delta} x^{\mu -}\bar{\delta} y_2^{\alpha_2 -}}\frac{\bar{\delta}}{\bar{\delta} y_1^{\alpha_1 -}} + \frac{\bar{\delta}^3 F_4}{\bar{\delta} x^{\mu -}\bar{\delta} y_1^{\alpha_1 -}\bar{\delta} y_2^{\alpha_2 -}}\Big] S_{(3)}(y_2,x';x,y_1). \tag{A.8}$$

(Higher-order terms in the derivatives are neglected.) The operators $\bar{\delta}/\bar{\delta} y_1^{\alpha_1 -}$ and $\bar{\delta}/\bar{\delta} y_2^{\alpha_2 -}$ acting on $S_{(2)}$ and $S_{(3)}$, respectively, can be replaced in terms of the corresponding equation of motion operators and delta functions and then integrations by parts are carried out; simailarly, the operator $\bar{\delta}/\bar{\delta} y_1^{\alpha_1 -}$ acting on $S_{(3)}$ can be replaced by $\partial/\partial y_1 - \bar{\delta}/\bar{\delta} y_1^{+}$ followed by an integration by parts. One finds at the end:

$$\frac{\bar{\delta} S(x,x')}{\bar{\delta} x^{\mu -}} = \frac{\bar{\delta} F_2(x',x)}{\bar{\delta} x^{\mu -}}\, S(x,x') - \frac{\bar{\delta}^2 F_3(x',x,y_1)}{\bar{\delta} x^{\mu -}\bar{\delta} y_1^{\alpha_1 +}}\, S_0(x,y_1)\,\gamma^{\alpha_1}\, S_{(2)}(y_1,x';x)$$
$$+\frac{\bar{\delta}^3 F_4(x',x,y_1,y_2)}{\bar{\delta} x^{\mu -}\bar{\delta} y_1^{\alpha_1 +}\bar{\delta} y_2^{\alpha_2 +}}\, S_0(x,y_1)\,\gamma^{\alpha_1}\, S_0(y_1,y_2)\,\gamma^{\alpha_2}\, S_{(3)}(y_2,x';x,y_1)$$
$$+\frac{\bar{\delta}^2 F_4}{\bar{\delta} x^{\mu -}\bar{\delta} y_2^{\alpha_2 +}}\,\frac{\bar{\delta} F_4}{\bar{\delta} y_1^{\alpha_1 +}} S_0(x,y_1)\,\gamma^{\alpha_1}\, S_0(y_1,y_2)\,\gamma^{\alpha_2}\, S_{(3)}(y_2,x';x,y_1). \tag{A.9}$$

This result could also have been obtained by using the method of Secs. 4 and 5, but using for the expansion of $S(A)$ Eq. (3.5), instead of (3.12). It is sufficient for this to replace in Eq. (5.8) $\bar{\delta} S/\bar{\delta} y_i$ by zero and S in internal lines by S_0.

If we had started for the previous calculations from representations (A.1) and (A.2), we would have found in a first stage the terms of Eq. (A.9) with a remainder containing difference terms of the type $(\bar{\delta}^2 F_4/\bar{\delta} x^- \bar{\delta} y_1^+ - \bar{\delta}^2 F_3/\bar{\delta} x^- \bar{\delta} y_1^+)\bar{\delta}/\bar{\delta} y_2^+ e^{F_4}$, $(\bar{\delta} F_4/\bar{\delta} x^- - \bar{\delta} F_3/\bar{\delta} x^-)\bar{\delta}^2/\bar{\delta} y_1^+ \bar{\delta} y_2^+ e^{F_4}$, etc. Integrations by parts show that the remainder is null. For this, one must proceed in two steps. First one converts $\bar{\delta}/\bar{\delta} y_2^+$ into $\bar{\delta}/\bar{\delta} y_2^-$ by the formula $\bar{\delta}/\bar{\delta} y_2^+ = \partial/\partial y_2 - \bar{\delta}/\bar{\delta} y_2^-$ and an integration by parts is done with respect to the total derivative of y_2. Second, for the term containing $\bar{\delta}/\bar{\delta} y_2^-$, one completes the exponential function with the multiplicative propagator $S_0(y_2,x')$ into $S_{(3)}$ and then the operator $\bar{\delta}/\bar{\delta} y_2^-$ is replaced in terms of the equation of motion operator and a delta function and a new integration by parts is done. The net result is zero. This method of calculation is repeated for all parts of the remainder. One thus finds the same result (A.9) from both representations (A.1)-(A.2) and (A.3)-(A.4).

Inspection of Eq. (A.9) shows that the last term is of the nested type, with $\bar{\delta}F_4/\bar{\delta}y_1^+$ representing a kind of self-energy insertion on the line xy_2. It should naturally be grouped with the free propagator S_0 appearing in front of $S_{(2)}$ (with a relabelling of the variables y_1 and y_2) to produce the full Green's function S [Eq. (A.1)] (at the present level of approximation). Nevertheless, we are faced with the phenomenon of the Wilson loop background effect: the various factors that appear in the last term of Eq. (A.9) involve F_4 and not F_3 which is the required function in the next-to-leading term of S. The recombinations that we can do to reconstruct full Green's functions in internal lines leave at the end remainders of the nested type.

Before proceeding to a recombination of the above factors in the general case, let us consider first the particular case of two-dimensional QCD in the quenched approximation (quark loops neglected) [13]. We assume that the Wilson loop contours that mainly contribute to the internal integrations are simple and convex, in particular without self-intersections. In that case, the logarithm of the Wilson loop average is given by the area of the surface delimited by the closed contour and the functions F_i are proportional to such areas. Furthermore, it is evident that these areas are separable into smaller ones. For the specific case above, we have $F_4(x', x, y_1, y_2) = F_3(x', x, y_2) + F_3(x, y_1, y_2)$, which also implies $\bar{\delta}F_4(x', x, y_1, y_2)/\bar{\delta}y_1^+ = \bar{\delta}F_3(x, y_1, y_2)/\bar{\delta}y_1^+$, $\bar{\delta}^2F_4(x', x, y_1, y_2)/\bar{\delta}x^-\bar{\delta}y_2^+ = \bar{\delta}^2F_3(x', x, y_2)/\bar{\delta}x^-\bar{\delta}y_2^+$, etc. With such decompositions, one easily transforms the last term of Eq. (A.9) into a form that is absorbed by the term containing $S_{(2)}$ to yield in front of it the full Green's function S. Replacing in the remaining term containing $S_{(3)}$ the free propagators S_0 by S (valid at the present level of approximation), one finds

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} = \frac{\bar{\delta}F_2(x',x)}{\bar{\delta}x^{\mu-}}\,S(x,x') - \frac{\bar{\delta}^2F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}}\,S(x,y_1)\,\gamma^{\alpha_1}\,S_{(2)}(y_1,x';x)$$
$$+\frac{\bar{\delta}^3F_4(x',x,y_1,y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}\bar{\delta}y_2^{\alpha_2+}}\,S(x,y_1)\,\gamma^{\alpha_1}\,S(y_1,y_2)\,\gamma^{\alpha_2}\,S_{(3)}(y_2,x';x,y_1), \qquad (A.10)$$

which is an expansion with full Green's functions in internal lines and kernels of the irreducible type without nested diagrams.

In four dimensions, the above decompositions of the Wilson loop averages are not generally valid and one has to evaluate the remainder with respect to Eq. (A.10). To this end, we complete in Eq. (A.9) the factor $S_0(x, y_1)$ in the term containing $S_{(2)}$ into $S(x, y_1)$ and isolate the rest, which is equal to

$$-\,S_0(x,z_1)\gamma^{\beta_1}S_0(z_1,y_1)\frac{\bar{\delta}F_3(y_1,x,z_1)}{\bar{\delta}z_1^{\beta_1+}}e^{F_3(y_1,x,z_1)}\gamma^{\alpha_1}\frac{\bar{\delta}^2F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}}S_{(2)}(y_1,x';x) \quad (A.11)$$

and which we write in the form

$$-S_0(x,z_1)\gamma^{\beta_1}S_0(z_1,y_1)\frac{\bar{\delta}F_3(y_1,x,z_1)}{\bar{\delta}z_1^{\beta_1+}}e^{F_3(y_1,x,z_1)}\gamma^{\alpha_2}\delta^4(y_1-y_2)$$
$$\times\frac{\bar{\delta}^2F_4(x',x,y_1,y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}}S_{(3)}(y_2,x';x,y_1). \qquad (A.12)$$

Writing the delta-function in the form $i\delta^4(y_1 - y_2) = (i\gamma.\partial_{(y_1)} - m)S_0(y_1, y_2)$ and making an integration by parts with respect to y_1 and neglecting higher-order derivative terms, we obtain

$$-S_0(x, y_1)\gamma^{\alpha_1} S_0(y_1, y_2)\frac{\bar{\delta}F_2(y_1, x)}{\bar{\delta}y_1^{\alpha_1+}}e^{F_2(y_1, x)}\gamma^{\alpha_2}\frac{\bar{\delta}^2F_4(x', x, y_1, y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}}S_{(3)}(y_2, x'; x, y_1). \quad \text{(A.13)}$$

In the first terms we recognize the dominant pieces of the product $\frac{\bar{\delta}S(x,y_1)}{\bar{\delta}y_1^{\alpha_1+}}\gamma^{\alpha_1} S(y_1, y_2)$. Using the equation of motion of S and making again an integration by parts with respect to y_1 (neglecting higher-order derivatives) we find the final expression

$$S(x, y_1)\gamma^{\alpha_1}\frac{\bar{\delta}S(y_1, y_2)}{\bar{\delta}y_1^{\alpha_1-}}\gamma^{\alpha_2}\frac{\bar{\delta}^2F_4(x', x, y_1, y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}}S_{(3)}(y_2, x'; x, y_1), \quad \text{(A.14)}$$

which, when grouped with the last term of Eq. (A.9), in which the S_0s may be replaced by Ss, yields the nested piece of Eq. (5.8) accompanying $S_{(3)}$:

$$\frac{\bar{\delta}^2F_4(x', x, y_1, y_2)}{\bar{\delta}x^{\mu-}\bar{\delta}y_2^{\alpha_2+}}S(x, y_1)\gamma^{\alpha_1}\Big(\frac{\bar{\delta}S(y_1, y_2)}{\bar{\delta}y_1^{\alpha_1-}} + S(y_1, y_2)\frac{\bar{\delta}F_4}{\bar{\delta}y_1^{\alpha_1+}}\Big)\gamma^{\alpha_2}S_{(3)}(y_2, x'; x, y_1). \quad \text{(A.15)}$$

We thus recover, together with the terms of Eq. (A.10), the first terms of Eq. (5.8), up to $S_{(3)}$. The calculation could be continued to higher-orders in the derivative terms, but it rapidly becomes complicated and loses interest for practical applicability. The method is useful for low-order perturbative calculations and for analysis of general qualitative properties.

B Quark self-energy

In order to construct the quark self-energy function directly, without having recourse to the explicit expression of the Green's function S, we start from its relationship with the derivative terms of S. The self-energy function, which we designate by Σ, is defined from the inverse of the Green's function:

$$iS^{-1}(x, x') = (i\gamma.\partial_{(x)} - m)\,\delta^4(x - x') - \Sigma(x, x'). \quad \text{(B.1)}$$

Comparison with the equations of motion (3.8) and (3.9) yields:

$$\Sigma(x, x'')\,S(x'', x') = i\gamma^\mu\,\frac{\bar{\delta}S(x, x')}{\bar{\delta}x^{\mu-}}, \qquad S(x, x'')\,\Sigma(x'', x') = -i\frac{\bar{\delta}S(x, x')}{\bar{\delta}x'^{\nu+}}\,\gamma^\nu. \quad \text{(B.2)}$$

Letting S^{-1} act on these equations, one ends up with the equation for Σ:

$$\Sigma(x, x') = -i\gamma^\mu\,\frac{\bar{\delta}^2S(x, x')}{\bar{\delta}x^{\mu-}\bar{\delta}x'^{\nu+}}\,\gamma^\nu + i\Sigma(x, y)\,S(y, y')\,\Sigma(y', x'). \quad \text{(B.3)}$$

The second-order derivative $\bar{\delta}^2 S(x,x')/\bar{\delta}x^{\mu-}\bar{\delta}x'^{\nu+}$ corresponds to a generalization of the first-order derivatives $\bar{\delta}S/\bar{\delta}x^{\pm}$, defined in Eqs. (2.20) and (2.6)-(2.7), where now the two derivations act on the same segment xx'; in this case, one has also to take into account the contributions coming from coincident points. Explicitly, one has:

$$\frac{\bar{\delta}^2 U(x',x)}{\bar{\delta}x^{\mu-}\bar{\delta}x'^{\nu+}} = (ig)^2 \int_0^1 d\lambda d\lambda'(1-\lambda)\lambda' \Big[U(1,\lambda')z'^{\beta}(\lambda')F_{\beta\nu}(z(\lambda'))U(\lambda',\lambda)$$
$$\times z'^{\alpha}(\lambda)F_{\alpha\mu}(z(\lambda))U(\lambda,0) + U(1,\lambda)z'^{\alpha}(\lambda)F_{\alpha\mu}(z(\lambda))U(\lambda,\lambda')z'^{\beta}(\lambda')F_{\beta\nu}(z(\lambda'))U(\lambda',0)\Big]$$
$$+ig\int_0^1 d\lambda(1-\lambda)\lambda\, U(1,\lambda)z'^{\alpha}(\lambda)\Big(\nabla_\nu F_{\alpha\mu}(z(\lambda))\Big)U(\lambda,0)$$
$$+ig\int_0^1 d\lambda(1-\lambda)\, U(1,\lambda)F_{\nu\mu}(z(\lambda))U(\lambda,0), \tag{B.4}$$

where $z'(\lambda) = \frac{\partial z}{\partial\lambda} = x' - x$ and ∇ is the covariant derivative, $(\nabla F) = (\partial F) + ig[A,F]$. The calculation was done by first deriving with respect to x and then with respect to x'. Had we interchanged the orders of derivation, the last two terms would be modified in the following way: $z'^{\alpha}\nabla_\nu F_{\alpha\mu} \rightarrow z'^{\alpha}\nabla_\mu F_{\alpha\nu}$, $(1-\lambda)F_{\nu\mu} \rightarrow -\lambda F_{\mu\nu}$. The two expressions are, however, equivalent, since the orders of derivation are irrelevant, due to the Bianchi identity satisfied by F. This can be checked directly by taking the difference of the two expressions.

The two derivative term (B.4) requires a careful treatment, since it contains divergences or singularities not present in one derivative terms. This is the case for the trace of the tensor (B.4); for neighboring points in the expression inside the brackets, the two Fs lead to a divergence, even when short-distance perturbative interactions are ignored [32, 26]; still for the trace, the term with the covariant derivative reduces to the gluon equation of motion operator and hence yields a delta-function (plus a quark current); these singular terms should be grouped together to set up a regularized form of the corresponding quantities. Concerning the traceless part of the coincident points contribution (the last two terms of Eq. (B.4)), we observe that it is of order g and not g^2; this implies that when expanding equation (B.3) in terms of derivatives of Wilson loop averages, one should count the latter contribution as a one derivative term. Finally, the role of the last term of Eq. (B.3) is to cancel similar reducible type terms that might emerge from the expansion of the second-order derivative piece. Nevertheless, because of existing background effects, as in the case of nested diagrams met in Sec. 5, the cancellations are only partial.

The above features make the direct treatment of the self-energy function rather intricate and less appealing than that of the Green's function itself. This underlines the fact that proper vertices do not seem to play a primary role in the present approach. We shall not pursue any longer here the study of the self-energy function.

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